

ON THE CHARACTERIZATION OF HILBERTIAN FIELDS

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ABSTRACT. The main goal of this work is to answer a question of Dèbes and Haran by relaxing the condition for Hilbertianity. Namely we prove that for a field K to be Hilbertian it suffices that K has the irreducible specialization property merely for *absolutely* irreducible polynomials.

1. INTRODUCTION

Let K be a number field. Hilbert's irreducibility theorem [4] gives for irreducible polynomials $f_i(T_1, \dots, T_r, X_1, \dots, X_s) \in K[\mathbf{T}, \mathbf{X}]$, $i = 1, \dots, n$ and a nonzero polynomial $p(\mathbf{T}) \in K[\mathbf{T}]$ an r -tuple $\mathbf{a} \in K^r$ for which $p(\mathbf{a}) \neq 0$ and all $f_i(\mathbf{a}, \mathbf{X})$ are irreducible in $K[\mathbf{X}]$. Actually, Hilbert's proof shows that it suffices to have a weaker irreducible specialization property, namely to have such an irreducible specialization only for one irreducible $f(T, X) \in K[T, X]$, separable in X and $p(T) \in K[T]$.

A field satisfying the latter property is called **Hilbertian**. So if K is Hilbertian, then the above stronger irreducibility specialization property holds, provided that $s = 1$ and $f_i(T_1, \dots, T_r, X)$ is separable in X , for each $i = 1, \dots, n$. Moreover to have irreducible specializations for any $s \geq 1$ and with no separability assumption, it is sufficient and necessary that K is Hilbertian and imperfect (Uchida's Theorem [7], see also [2, Proposition 12.4.3]).

Hilbert's irreducibility theorem has numerous applications in number theory (see e.g. [6]). In particular, Hilbert's original motivation for this theorem, the inverse Galois problem over a field K , which asks what finite groups occur as Galois group over K . Those applications make the question what fields are Hilbertian interesting.

The main goal of this paper is to answer a question of Dèbes and Haran [1] by proving that for a field K to be Hilbertian it suffices that K has the irreducible specialization property just for absolutely irreducible polynomials:

Theorem 1. *Let K be a field. Assume that for any absolutely irreducible $f(T, X) \in K[T, X]$, separable in X and any nonzero $p(T) \in K(T)$ there exists $a \in K$ such that $p(a) \neq 0$ and $f(a, X)$ is irreducible. Then K is Hilbertian.*

This theorem is known 'a priori' for special fields, namely PAC fields [1, Theorem 4.2]. A field K is **Pseudo Algebraically Closed (PAC)** if $V(K) \neq \emptyset$ for any

absolutely irreducible variety V defined over K . For PAC fields there is a connection between group theoretic properties of the absolute Galois group $\text{Gal}(K)$ and irreducible specializations of polynomials. We describe this connection below.

We prove Theorem 1 for an arbitrary field K . In fact, the argument we are using seems simpler than the argument used in [1] for the case where K is PAC.

In the rest of the introduction we describe the research that led Dèbes-Haran to their question and then briefly explain the main ingredient of the proof of Theorem 1.

The Hilbertianity property can be reformulated in terms of fields and places as follows (see Lemma 2): Let t be a transcendental element over a field K . Then K is Hilbertian if and only if the following property (*) holds for any finite separable extension $F/K(t)$ and nonzero $p(T) \in K[T]$.

(*) There exists a K -place ψ of F such that $a = \psi(t) \in K$, $p(a) \neq 0$, and the degree of ψ equals to the degree $[F : K(t)]$.

Here a **K -place** of a function field F/K is a place φ of F such that $\varphi(x) = x$ for all $x \in K$. The degree of φ is defined to be $\deg \varphi = [N : K]$, where N is the residue field of F under φ .

In [3] Fried and Völklein introduce the class of Regular-Galois-Hilbertian fields – An **RG-Hilbertian** field is a field which satisfies (*) for any finite Galois $F/K(t)$ for which F is regular over K and nonzero $p(T) \in K[T]$. This class of fields is important in the context of the inverse Galois problem. For example, considering a PAC field K of characteristic 0, they showed that K is RG-Hilbertian if and only if any finite group occurs as a Galois group over K and that K is Hilbertian if and only if K is ω -free (i.e. any finite embedding problem has a proper solution). These results are generalized for a field with an arbitrary characteristic by Pop [5].

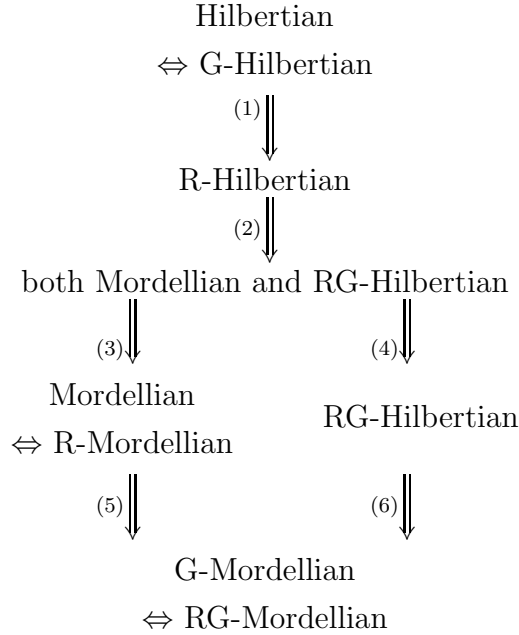
Using these group theoretic characterizations of Hilbertianity over a PAC field, Fried and Völklein give an example of a PAC field which is non-Hilbertian but is RG-Hilbertian, by constructing a projective profinite group having any finite group as a quotient, but some finite embedding problem is not solvable.

In [1] Dèbes and Haran construct some concrete new examples of non-Hilbertian RG-Hilbertian fields, which, in contrast to Fried-Völklein examples, are not PAC, and are even quite small over \mathbb{Q} in a certain sense. Also, they exhibit other variants of Hilbertianity which divide into two kinds. The first is consisted on the R-Hilbertian and G-Hilbertian fields, which satisfy (*) for any regular, resp. Galois, $F/K(t)$.

The second kind comes from the following characterization of Hilbertian fields. A necessary and sufficient condition for Hilbertianity is that for any irreducible

$f_1(T, X), \dots, f_r(T, X) \in K[T, X]$ that are separable and of degree > 1 in X and for any nonzero $p(T) \in K[T]$ there exists $a \in K$ for which $p(a) \neq 0$ and no $f_i(a, X)$ has a root in K [2, Lemma 13.1.2 and Proposition 13.2.2]. Then the class of *Mordellian* fields is defined – a field K is Mordellian if the above specialization property holds for one polynomial (i.e. $r = 1$). Similarly to the above are defined R-Mordellian, RG-Mordellian, and G-Mordellian fields.

Dèbes and Haran then sum up (using the following nice diagram) the connections between all the variants. Also none of the converses to (2), (3), (4), (5), and (6) in the diagram holds [1, Theorem 5.1]. In case K is PAC, using a sophisticated group theoretic construction, Dèbes and Haran show that the converse of (1) holds, but for an arbitrary K they simply say “*We do not know whether the converse of (1) holds in general...*” Theorem 1 then completes the job by showing that the converse of (1) always holds.



We conclude the introduction by a brief survey of the proof of Theorem 1. It is well known that it suffices to verify (*) in case $F/K(t)$ splits, i.e., we can assume $F = F_0L$, where $F_0/K(t)$ is regular and L/K Galois (see Lemma 3). A simple observation is that an irreducible specialization of F_0 gives an irreducible specialization of F if and only if the residue field of F_0 is linearly disjoint from L over K .

The main argument is to consider many copies of $F_0/K(t)$ and then to simultaneously irreducibly specialize all of them. Then if we have enough copies, at least

one of the specializations is ‘good,’ i.e., its residue field would be linearly disjoint from L .

Notation. Throughout the paper we use E, K, L, F to denote fields, T, X for variables, and t for a transcendental element over K . Bold face letters denote tuples, e.g., $\mathbf{T} = (T_1, \dots, T_r)$ (resp., $\mathbf{t} = (t_1, \dots, t_r)$) denotes a tuple of variables (resp., transcendental elements). As above, we say that an extension $F/K(\mathbf{t})$ is regular, if F is regular over K .

2. AUXILIARY LEMMAS

Lemma 2. *A field K is Hilbertian if and only if (*) holds for every finite separable extension $F/K(t)$ and nonzero $p(T) \in K[T]$.*

Proof. [2, Lemma 13.1.1] implies that a Hilbertian field K satisfies (*).

Conversely, let $f(T, X) \in K[T, X]$ be an irreducible polynomial that is separable in X and let $0 \neq p(T) \in K[T]$. Let $q(T)$ be the product of $p(T)$ with the leading coefficient of $f(T, X)$ and its discriminant (regarding f as a polynomial in X over $K(T)$). Let ψ be the corresponding K -place that (*) gives for $F/K(t)$ and $q(T)$, where $F = K(t)[X]/(f(t, X))$.

Then the residue field N of F under ψ is generated by a root of $f(a, X)$ [2, Remark 6.1.7]. Thus $[F : K(t)] = [N : K]$ implies that $f(a, X)$ is irreducible. \square

The following observation gives a sufficient condition for a polynomial to have an irreducible specialization in terms of a place of a regular extension having certain properties.

Lemma 3. *Let $f(T, X) \in K[T, X]$ be an irreducible polynomial that is separable in X . Then there exists a nonzero $p(T) \in K[T]$, a Galois extension L/K , and a separable regular extension $F/K(t)$ such that the following holds. Let ψ be a K -place of F with residue field N . Assume that $a = \psi(t) \in K$, $p(a) \neq 0$, $[N : K] = [F : K(t)]$, and N is linearly disjoint from L over K . Then $f(a, X)$ is irreducible.*

Proof. Let x be a root of $f(t, X)$ in a separable closure of $K(t)$. By [2, Lemma 13.1.3] there exist fields F and L such that F/K is regular and $t \in F$, L/K is Galois, $x \in FL$ and $FL/K(t)$ is Galois. Let $E = FL$ and let $p(t)$ be the product of the leading coefficient of $f(t, X)$ and its discriminant as a polynomial in X .

It suffices to find a K -place φ of E such that $a = \varphi(t) \in K$, $p(a) \neq 0$, $\deg \varphi = [E : K(t)]$ (w.r.t. E/K). Indeed, assume φ is such a place and let M denote the residue field of E . Since $p(a) \neq 0$ we have that $b = \varphi(x)$ is finite. Hence the residue

field of $K(t, x)$ under φ is $K(b)$. Since $f(a, b) = 0$ we have

$$\begin{aligned} \deg f(a, X) &\geq [K(b) : K] = \frac{[M : K]}{[M : K(b)]} = \frac{[E : K(t)]}{[M : K(b)]} \geq \frac{[E : K(t)]}{[E : K(t, x)]} \\ &= [K(t, x) : K(t)] = \deg_X f(t, X) = \deg f(a, X). \end{aligned}$$

Therefore $\deg f(a, X) = [K(b) : K]$ which implies that $f(a, X)$ is irreducible, as needed.

$$\begin{array}{ccc} F & \xrightarrow{\quad} & E \\ \downarrow & \nearrow_{K(t,x)} & \downarrow \\ K(t) & \xrightarrow{\quad} & L(t) \end{array} \quad \xrightarrow{\varphi} \quad \begin{array}{ccc} N & \xrightarrow{\quad} & M \\ \downarrow & \nearrow_{K(b)} & \downarrow \\ K & \xrightarrow{\quad} & L \end{array}$$

Let ψ be the K -place of F given by the assumption. Extend ψ to an L -place φ of E . Let M, N be the respective residue fields of E, F under φ . Then as $E = FL$ and φ is an L -place we have that $M = NL$. Since N and L are linearly disjoint over K , F and $L(t)$ are linearly disjoint over $K(t)$, and $\deg \psi = [F : K(t)]$ it follows that

$$[M : K] = [N : K][L : K] = [F : K(t)][L : K] = [E : K(t)].$$

Finally, since $\psi(t) = \varphi(t)$, we have $p(\varphi(t)) \neq 0$, and thus the assertion follows. \square

A similar argument gives the following result.

Lemma 4. *Let F_1, \dots, F_r be linearly disjoint separable extensions of a field E and let $F = F_1 \cdots F_r$. Let φ be a place of F/E with a residue field extension N/K and of degree $\deg \varphi = [F : E]$. Let N_i be the residue field of E_i under φ , for each $i = 1, \dots, r$. Then $[N_i : K] = [F_i : E]$ and N_1, \dots, N_r are linearly disjoint over K .*

Proof. Let F_0 be a subextension of F/E with residue field N_0 . As $[F : E] = [N : K]$ we have

$$[F_0 : E] = [F : E]/[F : F_0] = [N : K]/[F : F_0] \leq [N : K]/[N : N_0] = [N_0 : K] \leq [F_0 : E],$$

and hence $[N_0 : K] = [F_0 : E]$. In particular, for $F_0 = F_i$ we get $[N_i : K] = [F_i : E]$.

Next take $F_0 = F$. Then $N_0 = N = N_1 \cdots N_r$, and we have

$$[N_1 \cdots N_r : K] = [N : K] = [F : E] = [F_1 : E] \cdots [F_r : E] = [N_1 : K] \cdots [N_r : K],$$

which implies that N_1, \dots, N_r are linearly disjoint over K . \square

The following well known consequence of Bertini-Noether lemma and Matsusaka-Zariski theorem reduces the transcendence degree of a regular extension to 1 (cf. proof of [2, Proposition 13.2.1]). For the sake of completeness the proof is included below.

Lemma 5. *Let $r \geq 2$, let $(t, \mathbf{t}) = (t, t_1, \dots, t_r)$ be an $(r+1)$ -tuple of algebraically independent transcendental elements over an infinite field K , and let $E/K(\mathbf{t})$ be a finite regular separable extension. Then there exist $\alpha_i, \beta_i \in K$, $\beta_i \neq 0$, $i = 2, \dots, r$ for which the specialization $\mathbf{t} \mapsto (t, \alpha_2 + \beta_2 t, \dots, \alpha_r + \beta_r t)$ extends to a K -place φ of E with a regular residue field extension of degree $\deg \varphi = [E : K(\mathbf{t})]$.*

Proof. Let $x \in E$ be integral over $K[\mathbf{t}]$ such that $E = K(\mathbf{t}, x)$. Let $f(\mathbf{T}, X) \in K[\mathbf{T}, X]$ be the absolutely irreducible, separable and monic in X polynomial for which $f(\mathbf{t}, x) = 0$ and let $p(\mathbf{T})$ be its discriminant.

Take two variables U, V . Matsusaka-Zariski Theorem implies that $f(T_1, \dots, T_{r-1}, U + VT_1, X)$ is irreducible in the ring $\tilde{L}[T_1, \dots, T_{r-1}, X]$, where \tilde{L} is an algebraic closure of $K(U, V)$ [2, Proposition 10.5.4]. By Bertini-Neother Lemma there exists a nonzero $c(U, V) \in K[U, V]$ such that for any $\alpha_r, \beta_r \in K$ satisfying $c(\alpha_r, \beta_r) \neq 0$ the polynomial $f(T_1, \dots, T_{r-1}, \alpha_r + \beta_r T_1, X)$ remains absolutely irreducible over K [2, Proposition 10.4.2]. Since K is infinite, there exist $\alpha_r, \beta_r \in K$, $\beta_r \neq 0$ such that $c(\alpha_r, \beta_r) \neq 0$ and $p(T_1, \dots, T_{r-1}, \alpha_r + \beta_r T_1) \neq 0$. Induction on r yields $\alpha_i, \beta_i \in K$, $\beta_i \neq 0$, $i = 2, \dots, r$ such that $g(T, X) = f(T, \alpha_2 + \beta_2 T, \dots, \alpha_r + \beta_r T, X)$ is an absolutely irreducible polynomial and $q(T) = p(T, \alpha_2 + \beta_2 T, \dots, \alpha_r + \beta_r T) \neq 0$.

To conclude the proof, extend the specialization $\mathbf{t} \mapsto (t, \alpha_2 + \beta_2 t, \dots, \alpha_r + \beta_r t)$ to a K -place φ of E with a residue field extension $E'/K(t)$ [2, Lemma 2.2.7]. Then $E' = K(t, x')$ where $x' = \varphi(x)$ is a root of $g(t, X)$; hence E' is regular over K and

$$\deg \varphi = [E' : K(t)] = \deg_X g(t, X) = \deg_X f(\mathbf{t}, X) = [E : K(\mathbf{t})],$$

as needed. \square

For a field extension L/K we set $s(L/K)$ to be the number of subextensions $K \subseteq L_0 \subseteq L$. Note that if L/K is finite and separable, then $s(L/K)$ is also finite.

Lemma 6. *Let L/K be a finite separable extension with Galois closure E/K . Let $r \geq s(E/K)$ and let $N = N_1 \cdots N_r$ be the compositum of linearly disjoint extensions N_1, \dots, N_r of K . Then there exists $i \in \{1, \dots, r\}$ for which N_i is linearly disjoint from L over K .*

Proof. It suffices to show that there exists $i \in \{1, \dots, r\}$ for which N_i is linearly disjoint from E over K . Let $E_i = N_i \cap E$. As E/K is Galois, N_i and E are linearly disjoint if and only if $E_i = K$. Assume thus that $E_i \neq K$ for all i . Since $r > s(E/K) - 1$, the pigeonhole principle gives $i \neq j$ for which $E_i = E_j$. But $E_i \cap E_j \subseteq N_i \cap N_j = K$, which implies that $E_i = K$, a contradiction. \square

3. PROOF OF THEOREM 1

We assume that for any absolutely irreducible polynomial $g(T, X) \in K[T, X]$ and nonzero $p(T)$ there exists $a \in K$ such that $p(a) \neq 0$ and $g(a, X)$ is irreducible. This assumption implies (*) for regular F/K and nonzero $p(T)$.

Let $F/K(t)$ be a separable extension of degree n with F/K regular and let L/K be a Galois extension. By Lemma 3, it suffices to show that there exists a K -place ψ of F satisfying

- (1) $a = \psi(t) \in K$
- (2) $p(a) \neq 0$
- (3) $[N : K] = n$
- (4) N and L are linearly disjoint over K ,

where N is the residue field of F under ψ .

Let $r \geq s(L/K)$ (recall that $s(L/K)$ is the number of subextensions of L/K). Take r algebraically disjoint copies of $F/K(t)$, that is to say, consider an r -tuple $\mathbf{t} = (t_1, \dots, t_r)$ of algebraically independent transcendental elements and for each $i = 1, \dots, r$ consider an extension $F_i/K(t_i)$ and a K -isomorphism $\nu_i: F \rightarrow F_i$ under which t maps to t_i . Let $E_i = F_i K(\mathbf{t})$ and $E = E_1 \cdots E_r$. Then E_1, \dots, E_r are linearly disjoint over $K(\mathbf{t})$ and we have

$$[E_i : K(\mathbf{t})] = [F_i : K(t_i)] = n,$$

for $i = 1, \dots, r$, and thus $n^r = [E_1 : K(\mathbf{t})] \cdots [E_r : K(\mathbf{t})] = [E : K(\mathbf{t})]$.

Lemma 5 gives $\alpha_i, \beta_i \in K$, $\beta_i \neq 0$ for which the specialization $\mathbf{t} \mapsto (t_0, \alpha_2 + \beta_2 t_0, \dots, \alpha_r + \beta_r t_0)$ (with transcendental element t_0) extends to a K -place φ of $E/K(\mathbf{t})$ with a regular residue field $E'/K(t_0)$ of degree $\deg \varphi = [E : K(\mathbf{t})] = n^r$. Let E'_i be the residue field of E_i under φ , $i = 1, \dots, r$.

Note that the set $A_0 = \{a_0 \in K \mid \exists 1 \leq i \leq r \text{ such that } p(\alpha_i + \beta_i a_0) = 0\}$ is finite, since $\beta_i \neq 0$, and hence $q(T) = \prod_{a_0 \in A_0} (T - a_0)$ is a polynomial. Therefore we can apply (*) for the regular extension $E'/K(t_0)$ and $q(T) \in K[T]$ to get a place ψ' of E' of degree n^r such that $a_0 = \psi'(t_0) \in K$ and $q(a_0) \neq 0$.

Let N_i be the residue field of E'_i under ψ' , $i = 1, \dots, r$. In this setting, Lemma 4 asserts that $[N_i : K] = [E'_i : K(t_0)] = [E_i : K(\mathbf{t})] = n$ and N_1, \dots, N_r are linearly disjoint over K . Since $r \geq s(L/K)$, Lemma 6 gives $i \in \{1, \dots, r\}$ for which N_i and L are linearly disjoint over K .

$$\begin{array}{ccccccccc}
& & & \psi & & & & & \\
F & \xrightarrow{\nu_i} & F'_i & \xrightarrow{\quad} & E_i & \xrightarrow{\varphi_i} & E'_i & \xrightarrow{\psi'} & N_i \\
| & & | & & | & & | & & | \\
K(t) & \longrightarrow & K(\mathbf{t}) & \hookrightarrow & K(\mathbf{t}) & \longrightarrow & K(t_0) & \longrightarrow & K \\
t & \xrightarrow{\nu_i} & t_i & \longmapsto & t_i & \longmapsto & \alpha_i + \beta_i t_0 & \longmapsto & \alpha_i + \beta_i a_0
\end{array}$$

To conclude the proof set $\psi = \psi' \varphi_i \nu_i$, where $\varphi_i = \varphi|_{E_i}$ is the restriction of φ to E_i . We have

$$a = \psi(t) = \psi'(\varphi(t_i)) = \psi'(\alpha_i + \beta_i t_0) = \alpha_i + \beta_i a_0 \in K,$$

and hence (1) holds. The residue field of F under ψ is N_i ; this proves (3) and (4). Recall that $q(a_0) \neq 0$ implies that $p(a) \neq 0$, hence we have (2) and the proof of Theorem 1 is completed.

Remark 7 (Hilbert sets). Let $f(\mathbf{T}, X) \in K[\mathbf{T}, X]$ be an irreducible polynomial that is separable in X and $p(\mathbf{T}) \in K[T]$. Then the corresponding **Hilbert set** is defined to be

$$H_K(f; p) = \{\mathbf{a} \in K^r \mid f(\mathbf{a}, X) \text{ is irreducible and } q(\mathbf{a}) \neq 0\} \subseteq K^r.$$

The proof of Theorem 1 gives the following assertion. Let $f(T, X) = f_0(T)X^d + \dots \in K[T, X]$ be an irreducible polynomial that is separable in X and let $p(T) \in K[T]$ be nonzero. Then there exists an absolutely irreducible polynomial $g(T_1, \dots, T_r, X) \in K[\mathbf{T}, X]$ (for any sufficiently large r) and $q(\mathbf{T}) \in K[\mathbf{T}]$ such that the following holds. For any $\mathbf{a} \in H_K(g; q)$ there exists $i \in \{1, \dots, r\}$ such that $a_i \in H_K(f; p)$.

(The above assertion follows from the proof of Theorem 1 by taking $g(\mathbf{T}, X)$ such that a root of $g(\mathbf{t}, X)$ generates $E/K(\mathbf{t})$ and $q(\mathbf{T}) = \prod_{i=1}^r p(T_i)f_0(T_i)$.)

The above remark leads to a slightly finer question than Dèbes-Haran question, which might be of interest:

Problem 8 (cf. [2, Problem 13.1.5]). Let $f(T, X) \in K[T, X]$ be an irreducible polynomial, separable in X and let $p(T) \in K[T]$ be nonzero. Does there exist an absolutely irreducible $g(T, X)$ and nonzero $q(T)$ such that $H_K(g; q) \subseteq H_K(f; p)$?

4. SMALL EXTENSIONS OF HILBERTIAN FIELDS

Definition 9. An extension M/K is called **small** if M/K is Galois and for every positive integer n there exists finitely many subextensions L of M/K of degree $\leq n$.

Example 10. If $\text{Gal}(M/K)$ is finitely generated, then $\text{Gal}(M/K)$ is small [2, Lemma 16.10.2].

Example 11. If $\text{Gal}(M/K) \cong \prod_p \mathbb{Z}_p^p$, where p runs over all primes and \mathbb{Z}_p is the p -adic group, then M/K is small and $\text{Gal}(M/K)$ is not finitely generated [2, Example 16.10.4].

We reprove [2, Proposition 16.11.1] using Theorem 1.

Theorem 12. *Let K be a Hilbertian field and M/K a small extension. Then M is Hilbertian.*

Proof. Let $f(T, X) \in M[T, X]$ be an absolutely irreducible polynomial that is separable and of degree n in X . Let K' be a finite subextension of M/K such that $f(T, X) \in K'[T, X]$. Then K' is Hilbertian [2, Corollary 12.2.3]. Evidently M/K' is also small. Let $r > \#$ of subextensions of M/K' of degree $\leq n$.

Let $F = K'(t)[X]/(f(t, X))$. Then F is regular over K' . Now we proceed as in the proof of Theorem 1: Take r copies $F_1/K'(t_1), \dots, F_r/K'(t_r)$ of $F/K(t)$. From the Hilbertianity of K' we get a specialization $(t_1, \dots, t_r) \mapsto (a_1, \dots, a_r) \in K'^r$ such that the residue fields N_i of F_i are linearly disjoint, and $[N_i : K] = [F_i : K(t_i)] = n$.

Now, if there exists $1 \leq i \leq r$ such that $N_i \cap M = K'$, then $[N_i M : M] = [N_i : K'] = n$. But $N_i M$ is the residue field of $F_i M$; so we are done.

Assume that $L_i := N_i \cap M \neq K'$ for all $1 \leq i \leq r$. Then, since $N_i \cap N_j = K'$ we have $L_i \cap L_j = K'$. In particular, L_1, \dots, L_r are distinct subextensions of M/K' of degrees $\leq n$. This contradicts the choice of r . \square

Remark 13. Actually a stronger statement than we proved is true, namely M is Hilbertian over K . Indeed, the fact that K' is not only Hilbertian, but also Hilbertian over K [2, Corollary 12.2.3] implies that we can choose a_i to be in K .

This stronger assertion is also proved in [2, Proposition 16.11.1].

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